2023

MATHEMATICS — HONOURS

Paper : DSE-A-1.1, DSE-A-1.2 and DSE-A-1.3

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations have usual meanings.

Paper : DSE-A-1.1

(Advanced Algebra)

Full Marks : 65

Group - A

(Marks : 20)

1. Answer ***all*** questions. In each question **one** mark is reserved for selecting the correct option and **one** mark is reserved for justification : (1+1)×10

(a) Let G be a group of order 231. Then the number of Sylow 7-subgroups of G is

(i) 2 (ii) 1

(iii) 7 (iv) 3.

(b) Let G be a non-commutative group of order p^3 where p is a prime. Then number of elements in the centre $Z(G)$ of G is

(i) p^3 (ii) p^2

(iii) p (iv) 1.

(c) In an integral domain

(i) every prime element is irreducible

(ii) a prime element may not be irreducible

(iii) every prime element is reducible

(iv) every element is irreducible.

(d) The ring $(\mathbb{Z}, +, \cdot)$ is

(i) an integral domain and a regular ring

(ii) an integral domain but not a regular ring

(iii) a regular ring but not a field

(iv) a regular ring but not an integral domain.

(e) The conjugacy class of (1 2 3) in S_3 is

(i) {(1 2 3), (1 2)}
(ii) {(1 2 3), (1 3)}
(iii) {(1 2 3), (1 3 2)}
(iv) {(1 2 3), (1 2), (1 3 2)}.

(f) Every commutative group of order 36 contains an element of order

(i) 2
(ii) 3
(iii) 4
(iv) 6.

(g) Which of the following group is not simple?

(i) Group of order 20
(ii) Group of order 21
(iii) Group of order 32
(iv) Group of order 12.

(h) If K is a field, then $K[x]$ is

(i) Integral Domain
(ii) Euclidean Domain
(iii) Principal Ideal Domain (PID)
(iv) Not PID.

(i) Which one of the following statements is true for a group of order 125?

(i) G is a simple group
(ii) G has a non-trivial centre
(iii) G is commutative
(iv) G is cyclic.

(j) Which is true for the ring $(\mathbb{Z}_{12}, +, \cdot)$?

(i) $\overline{3}$ is a prime element
(ii) $\overline{3}$ is an irreducible element
(iii) $\overline{5}$ is a prime element
(iv) $\overline{5}$ is an irreducible element.

Group - B

(Marks : 15)

2. Answer *any three* questions :

(a) Show that A_5 is a simple group. 5

(b) State and prove Sylow's First theorem. 1+4

(c) Let G be a group of order p^n, p a prime, and $n \in Z$, $n \geq 1$. Prove that any subgroup of G of order p^{n-1} is normal in G. 5

(d) Let G be a group and S be a G-set. If S is finite, then prove that

$$|S| = \sum_{a \in A} [G : G_a],$$

where A is the subset containing exactly one element from each orbit $[a]$. 5

(e) Let G be a group of order 99. Prove that G has a unique normal subgroup H of order 11. Also show that $H \subset Z(G)$. 2+3

Group - C

(Marks : 30)

3. Answer ***any six*** questions :

(a) For any field F, prove that the polynomial ring $F[x]$ is a principal ideal domain. Is $Z[x]$ a principal ideal domain? Justify your answer. 3+2

(b) In a principal ideal domain D, prove that a non-null ideal $\langle p \rangle$ is maximal if and only if p is an irreducible element in D. 2+3

(c) Prove that $Z[i]$ is an Euclidean domain. 5

(d) (i) Prove that the centre of a regular ring is regular.

(ii) If an integral domain D is regular, then prove that D is a field. 3+2

(e) Let R be a unique factorizable domain. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$ is a monic polynomial in $R[x]$ and $p \in R$ is a prime such that $p | a_{n-1}, p | a_{n-2}, \ldots, p | a_0$ but $p^2 \nmid a_0$. Prove that $f(x)$ is irreducible in $R[x]$. 5

(f) Justify the following statement by citing an example :

'There exists an integral domain where greatest common divisor of two elements may not exist'. 5

(g) (i) Let R be an integral domain and p be a prime element of R. Then prove that p is irreducible.

(ii) Prove that the converse is not true. 3+2

(h) Prove that a Factorisation Domain (FD) D is a Unique Factorisation Domain (UFD) iff every irreducible element of D is prime. 5

(i) Show that $\gcd\left(2, 1+i\sqrt{5}\right) = 1$ in the integral domain $Z[i\sqrt{5}]$. 5

(j) Define regular ring. Let R be a regular ring with more than one element. Suppose for all $x \in R$, there exists unique $y \in R$ such that $x = xyx$. Show that R is a ring with unity having no divisor of zero. 5

Paper : DSE-A-1.2

(Bio-Mathematics)

Full Marks : 65

The figures in the margin indicate full marks.

Group - A

(Marks : 20)

1. Answer the following multiple choice questions with only one correct option. Choose the correct option with proper justification. (1+1)×10

(a) In the following model :

$$\frac{1}{N}\frac{dN}{dt}=r\left[1-\left(\frac{N}{K}\right)^{\theta}\right];\, r, K, \theta \text{ being positive parameters,}$$

the steady state $N^*=0$ is

(i) unstable (ii) stable but not asymptotically stable

(iii) asymptotically stable (iv) none of these.

(b) In Gompertz growth model $\frac{dP}{dt}=CP\ln(K/P)$, the population (P) grows fastest when P is equal to

(i) 0 (ii) K

(iii) e/K (iv) K/e

C, K being positive parameters.

(c) In the following harvesting model :

$$\frac{dN}{dt}=rN\left(1-\frac{N}{K}\right)-qEN,$$

where the symbols have their usual meanings, the non-trivial steady state exists if

(i) $\frac{qE}{r}=1$ (ii) $\frac{qE}{r}>1$

(iii) $\frac{qE}{r}<1$ (iv) $\frac{q}{Er}<1$.

(d) The system $\frac{dx}{dt}=\mu x+x^2$, where $\mu \in \mathbb{R}$ is a parameter, has a

(i) pitchfork bifurcation (ii) saddle node bifurcation

(iii) transcritical bifurcation (iv) none of these.

(e) If α, β, γ, δ are positive parameters, the nullclines of the two-dimensional system

$$\frac{dx}{dt} = x(\alpha - \beta y),$$
$$\frac{dy}{dt} = y(-\gamma + \delta x),$$

intersect at

(i) $(0, 0), \left(\frac{\beta}{\alpha}, \frac{\gamma}{\delta}\right)$ (ii) $(0, 0), \left(\frac{\alpha}{\beta}, \frac{\delta}{\gamma}\right)$

(iii) $(0, 0), \left(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right)$ (iv) $(0, 0), \left(\frac{\beta}{\alpha}, \frac{\delta}{\gamma}\right)$.

(f) The steady state (0, 0) of the following system

$$\frac{dx}{dt} = 3x + 2y,$$
$$\frac{dy}{dt} = 4x + y,$$

is

(i) a centre (ii) a saddle point

(iii) a stable spiral (iv) an unstable spiral.

(g) The steady state (0, 0) of the system

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = (\alpha - 1)x - \alpha y,$$

where α is a positive parameter and $\alpha \neq 1$, is asymptotically stable if

(i) $0 < \alpha < 1$ (ii) $\alpha > 1$

(iii) $2 < \alpha < 3$ (iv) none of these.

(h) For the Kermack-McKendrick SIR model, S + I + R is equal to

(i) 2S (ii) 3I

(iii) 4R (iv) constant.

(i) The non-zero steady state of the logistic difference equation $x_{n+1} = rx_n(1 - x_n)$, $0 \leq r \leq 4$ is asymptotically stable if

(i) $0 < r < 1$ (ii) $1 < r < 3$

(iii) $0 \leq r < 3$ (iv) $3 \leq r < 4$.

(j) The difference equation $x_{n+1} = \frac{\alpha x_n}{1+x_n}$ ($\alpha \in \mathbb{R}$ being a parameter), has two distinct non-negative steady states if

(i) $\alpha > 0$ (ii) $\alpha > -1$

(iii) $\alpha < 1$ (iv) $\alpha > 1$.

Group - B

Unit - I

(Marks : 15)

Answer ***any one*** question.

2. (a) What is meant by Allee effect? Consider the growth model $\frac{dN}{dt} = rN\left(\frac{N}{A}-1\right)\left(1-\frac{N}{K}\right)$, where r, A, K are positive parameters and $A < K$. Determine the steady states and discuss their stability.

(b) Discuss the stability of the steady states of the following harvesting model :

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - h, \text{ for the cases } h <=> \frac{rK}{4},$$

where $r(>0)$ is the growth rate, $h(>0)$ is the constant rate harvesting and $K(>0)$ is the carrying capacity.

(c) Suppose that a population follows Malthus growth model. If it has 24×10^5 members after 5 years and 15×10^5 members after 15 years, what was the initial population size?

(2+2+3)+5+3

3. (a) What is meant by bifurcation? Discuss the saddle node bifurcation for the system

$$\frac{dx}{dt} = \mu + x^2, \ \mu \in \mathbb{R},$$

where μ is a parameter.

(b) Show by using the transformation $N = ax, T = \frac{a}{m}t$ the following spruce-budworm model :

$$\frac{dN}{dT} = rN\left(1-\frac{N}{K}\right) - \frac{mN^2}{a^2+N^2} \text{ (where the symbols have their usual meanings),}$$

can be put in the following dimensionless form

$$\frac{dx}{dt} = px\left(1-\frac{x}{q}\right) - \frac{x^2}{1+x^2},$$

where the constants p and q are to be determined by you. Also discuss the stability of the trivial steady state.

(c) For the logistic model $\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$ with $x(0) < \frac{K}{2}$ (where symbols have their usual meanings), show that $x = K$ is an asymptotically stable steady state. Also show that the solution curve has a point of inflexion at $x = \frac{K}{2}$. Hence draw the rough sketch of the solution curve.

(1+3)+{2+(1+1)+2}+(2+2+1)

Unit - II

(Marks : 20)

Answer *any two* questions.

4. (a) Linearise the classical Lotka-Volterra model

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = -\gamma y + \delta xy,$$

about the positive steady state, where $\alpha, \beta, \gamma, \delta$ are positive parameters. Hence show that both prey and predator have periodic solutions with same period.

(b) Investigate the stability of the non-trivial steady states of the following competition model :

$$\frac{dx}{dt} = x(16 - 2x - y),$$
$$\frac{dy}{dt} = y(12 - x - y).$$

(2+2+1+1)+4

5. Discuss the stability of the steady states of the following predator-prey system : 10

$$\frac{dx}{dt} = x\left(1 - \frac{x}{30}\right) - \frac{xy}{x+10},$$
$$\frac{dy}{dt} = -\frac{3y}{5} + \frac{xy}{x+10}.$$

6. Consider the following model of bacterial growth in a chemostat :

$$\frac{dN}{dt} = \left(\frac{k_1 C}{k_2 + C}\right)N - \frac{FN}{V},$$
$$\frac{dC}{dt} = -\alpha\left(\frac{k_1 C}{k_2 + C}\right)N - \frac{FC}{V} + \frac{FC_0}{V},$$

where the symbols have their usual meanings.

(a) Show that the equations can be reduced to the following dimensionless form by the substitution $N = \frac{Fk_2}{\alpha V k_1} u, \; C = k_2 v, \; t = \frac{V}{F}\tau$:

$$\frac{du}{d\tau} = \alpha_1 \left(\frac{v}{1+v}\right) u - u,$$

$$\frac{dv}{d\tau} = -\left(\frac{v}{1+v}\right) u - v + \alpha_2,$$

where α_1 and α_2 are the parameters to be determined by you.

(b) Find the steady states of the dimensionless system. Find the conditions on α_1 and α_2 so that the steady states become biologically meaningful.

(c) Determine the stability of the biologically meaningful steady states. 2+3+5

7. (a) Show that the following system :

$$\frac{dx}{dt} = -y + x\left(1 - x^2 - y^2\right),$$

$$\frac{dy}{dt} = x + y\left(1 - x^2 - y^2\right),$$

has a stable limit cycle.

(b) State the basic assumptions of the Kermack-McKendrick SIR model. Draw the flowchart and write the model equations. Find the basic reproduction number. 5+(2+2+1)

Unit - III

(Marks : 10)

Answer *any one* question.

8. (a) Suppose x^* is a steady state of the system $x_{n+1} = f(x_n)$, where $f(x)$ is a continuously differentiable function and $|f'(x^*)| \neq 1$. Prove that x^* is asymptotically stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$.

(b) Discuss the stability of each steady state of the system $x_{n+1} = \frac{3x_n}{2 + x_n}$ using cobweb diagram with initial value x_0. [Show at least three iterations in each case.] 5+(3+2)

9. (a) Consider the discrete-time predator-prey system :

$$x_{n+1} = ax_n\left(1 - x_n\right) - bx_n y_n,$$
$$y_{n+1} = -cy_n + dx_n y_n,$$

where a, b, c, d are positive parameters. Find the steady states of the system and discuss their stability.

(b) Show that the following difference equation :

$$x_{n+1} = \frac{rx_n^2}{x_n^2 + A}, \quad (r, A \text{ being positive parameters}),$$

where r, A are positive parameters, has three distinct steady states if $r > 2\sqrt{A}$ and discuss their stability. (3+3)+4

Paper : DSE-A-1.3

(Industrial Mathematics)

Full Marks : 65

The figures in the margin indicate full marks.

1. Choose the correct answer with proper justification/explanation for each of the following multiple choice question. (**one** mark for each correct answer and **one** mark for justification.) 2×10

(a) If the 2×2 matrix X satisfies the equation $X\begin{pmatrix}4 & 7\\ 5 & 9\end{pmatrix}=\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}$, then $X=$

(i) $\begin{pmatrix}-6 & 4\\ 13 & -10\end{pmatrix}$ (ii) $\begin{pmatrix}-6 & 5\\ 13 & -10\end{pmatrix}$

(iii) $\begin{pmatrix}-6 & 4\\ 12 & -10\end{pmatrix}$ (iv) $\begin{pmatrix}-6 & 4\\ 13 & -1\end{pmatrix}$.

(b) The attenuation coefficient of an X-ray beam measures

(i) proportion of the photons absorbed by each millimeter of a substance when an X-ray passes through it.

(ii) wavelength of the X-ray

(iii) proportion of the photons which are not absorbed by a substance when an X-ray passes through it.

(iv) None of the above.

(c) If $l_{t,\theta}$ be the line through the point $(t\cos\theta,\ t\sin\theta)$ and perpendicular to the unit vector $\hat{n}=(\cos\theta, \sin\theta)$, then $x+y=\sqrt{2}$ is same as

(i) $l_{1,\frac{\pi}{2}}$ (ii) $l_{1,\frac{\pi}{4}}$

(iii) $l_{0,\frac{\pi}{2}}$ (iv) $l_{\sqrt{2},\frac{\pi}{4}}$.

(d) Suppose f_1 is the attenuation-coefficient function corresponding to a disc of radius ½ centred at the origin and with constant density 1. Then, for every line $l_{0,\theta}$ through the origin, the Radon transform $\mathcal{R}f_1(0, \theta)=$

(i) 0 (ii) 1

(iii) 2 (iv) ½.

(e) *Algebraic reconstruction techniques (ARTs)* are techniques for reconstructing images
- (i) that have no direct connection to the Radon inversion formula
- (ii) that are same as the Radon inversion formula
- (iii) that are connected to but not same as the Radon inversion formula
- (iv) None of the above.

(f) If a signal $x(t)$ has a Fourier transform $X(\omega)$ and $x(t)$ is an even real function of t, then $X(\omega)$ is
- (i) a real and even function
- (ii) a real and odd function
- (iii) an imaginary and even function
- (iv) an imaginary and odd function.

(g) $\sqrt[n]{i}+\sqrt[n]{-i}$ is equal to
- (i) $2\cos\frac{\pi}{2n}$
- (ii) $2\sin\frac{\pi}{2n}$
- (iii) 0
- (iv) $2i\cos\frac{\pi}{2n}$.

(h) The Fourier cosine transform of e^{-bx} is
- (i) $\sqrt{\frac{2}{\pi}}\frac{b^2}{p^2+b^2}$
- (ii) $\sqrt{\frac{2}{\pi}}\frac{p^2}{p^2+b^2}$
- (iii) $\sqrt{\frac{2}{\pi}}\frac{b}{p^2+b^2}$
- (iv) $\sqrt{\frac{2}{\pi}}\frac{p}{p^2+b^2}$.

(i) Back projection of the Radon transform of a function
- (i) always reproduces the original function in any domain
- (ii) does not necessarily reproduce the original function
- (iii) only reproduces the original function in the domain of unit circle
- (iv) none of the above is true.

(j) Let $f(x)=e^{-Ax^2}$, for some positive constant $A>0$. Then the Fourier transform of $f(x)$ is
- (i) $\sqrt{\frac{\pi}{2A}}e^{-\frac{\omega^2}{2A}}$
- (ii) $\sqrt{\frac{\pi}{A}}e^{-\frac{\omega^2}{A}}$
- (iii) $\sqrt{\frac{\pi}{A}}e^{-\frac{\omega^2}{2A}}$
- (iv) $\sqrt{\frac{\pi}{A}}e^{-\frac{\omega^2}{4A}}$.

Unit - I

2. Answer *any two* questions :

(a) Explain Inverse problem with a mathematical example. Why is it necessary in the science of imaging? 2+3

(b) Define CT scan and explain it by an example. 1+4

(c) Define the well-posedness of a mathematical problem. Give an example of an ill-posed problem. 3+2

(d) Find the complementary function and general solution of the differential equation

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{2x}+x^2.$$ 2+3

Unit - II

3. Answer *any two* questions :

(a) Explain direct problem and indirect problem with suitable example. 3+2

(b) Find the inverse of $f(x)=(x-3)^3-1$, $x\in\mathbb{R}$. Also find $f^{-1}(0)$ and $f^{-1}(7)$. 3+2

(c) Suppose the eigenvalues of $A=\begin{pmatrix} a & -b \\ -b & c+d \end{pmatrix}$ are 2 and 7 and the eigenvalues of $B=\begin{pmatrix} a & -b \\ -b & c \end{pmatrix}$ are 1 and 5. Find A and B. 5

(d) Given that under the influence of a central force, the radius vector joining the centre of force to the particle, sweeps out equal areas in equal times. Solve the following inverse problem :

If a particle orbits in a circle and the radius vector sweeps out equal areas in equal times, then the particle is attracted by a central force to the origin. 5

Unit - III

4. Answer *any one* question :

(a) State Beer's law on X-ray beam. Write its differential equation form. Establish the result

$$\int_{x_0}^{x_1} A(x)dx=\ln\left(\frac{I_0}{I_1}\right),$$

where $A(x)$ is the attenuation coefficient function and $I(x)$ is the intensity of the X-ray beam. 5

(b) An X-ray beam propagating in a medium is defined by $A(x)=\frac{1}{\theta}-\frac{k-1}{x}$, where θ, $k>0$ are real constants. Prove that the intensity of the X-ray beam is a gamma distribution and the normalization constant is equal to $\frac{1}{\theta^k\Gamma(k)}$. 3+2

Unit - IV

5. Answer *any one* question :

(a) Find the Radon transform on a line $\mathcal{L}_{t,\theta}$ of the function $f(x,y)$ which is defined in an ellipse :

$$f(x,y)=\begin{cases}1-\sqrt{x^2+0.5y^2}, & \text{if, } x^2+0.5y^2\le 1\\ 0, & \text{if, } x^2+0.5y^2>1\end{cases}$$ 5

(b) Write a short note on Shepp-Logan Mathematical phantom. 5

Unit - V

6. Answer *any one* question :

(a) (i) Let $h=h(t,\theta)$ be a function whose inputs are polar coordinates. Define the Back Projection function $\mathcal{B}h(x,y)$ of h at the point (x,y).

(ii) Prove that the Back Projection function $\mathcal{B}h(x,y)$ is a linear transformation. 2+3

(b) If $h(t,\theta)=t^2\cos^2\theta$, then find the back projection of $h(t,\theta)$ at the point $(1,2)$. 5

Unit - VI

7. Answer *any two* questions :

(a) (i) What is algebraic reconstruction technique or ART?

(ii) Find the Fourier transform of $f(x)=\begin{cases}1, & |x|\le a\\ 0, & |x|>a\end{cases}$. 2+3

(b) Show that the inverse Fourier Transform of an even function is a real-valued function and the inverse Fourier Transform of an odd function is a purely imaginary function. 5

(c) Prove that for suitable functions f and g

$$\mathcal{F}(f\cdot g)=\frac{1}{2\pi}(\mathcal{F}f)*(\mathcal{F}g),$$ where * denotes convolution. 5

(d) For the system of two lines $x_1-x_2=0$ and $x_1+x_2=5$ and the starting point $x^0=\begin{bmatrix}3\\1\end{bmatrix}$, apply Kaczmarz's method to compute $x^{0,1}$ and $x^{0,2}$. 5

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